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Stress recovery procedure for solving boundary value problems in the mechanics of a deformable solid by the finite element method☆

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ABSTRACT

A stress recovery procedure, based on the determination of the forces at the mesh points using a stiffness matrix obtained by the finite element method for the variational Lagrange equation, is described. The vectors of the forces reduced to the mesh points are constructed for the known stiffness matrices of the elements using the displacements at the mesh points found from the solution of the problem. On the other hand, these mesh point forces are determined in terms of the unknown forces distributed over the surface of an element and given shape functions. As a result, a system of Fredholm integral equations of the first kind is obtained, the solution of which gives these distributed forces. The stresses at the mesh points are determined for the values of these forces found on the surfaces of the finite element mesh (including at the mesh points) using the Cauchy relations, which relate the forces, stresses and the normal to the surface. The special features of the use of the stress recovery procedure are demonstrated for a plane problem in the linear theory of elasticity.

In the numerical realization of linear and non-linear boundary value problems of the mechanics of a deformable solid, their variational (weak) formulation is implemented in the majority of cases. The Lagrange formulation, which, among all the formulations, has the smallest number of variable parameters and is realized using the finite element method, is conventionally employed. This leads to quite a good approximation of the required displacements field, and to a considerably worse approximation of the stress field, which is associated with the differentiation of the mesh (piecewise-continuous) functions. This shows up especially strongly at the boundary of the domain.

Well-known stress recovery methods (according to the terminology adopted in Ref. 1) are used to obtain a good approximation of the stress field within the limits of the Lagrange formulation: the method of averaging of the stresses over the elements adjacent to a given mesh point [2] with subsequent extrapolation of the values obtained onto the boundary of the domain, the method of conjugate approximation,[3] the construction of spline functions for the displacement field [4] and the use of fundamental solutions of a corresponding problem in mechanics.[5] These methods are not equivalent either in accuracy (for the one and the same order of the coordinate functions and the number of finite elements) or in the complexity of their numerical implementation. The method of averaging [2] is the least accurate but the simplest to implement. The remaining methods are considerably more accurate, but, in their implementation, they are equivalent to (or even more difficult than) the solution of the initial problem, that is, the determination of the displacements from the variational equation by the finite element method.

Adaptive algorithms,[6,7] which are either associated with the optimal arrangement of the finite element mesh points in a region without increasing their number and the degree of approximation (R-convergence) or with an increase in the number of elements, adapting their shape and arrangement to the problem (h-convergence) or with an increase in the degree of approximation of the functions in an element (p-convergence).

A good approximation of the stress field can be obtained when both the displacements field and the stress field are independently determined from the variational formulation of the problem, that is, when the Hu–Washizu or Reissner variational formulations are used. However, in this case, there is an increase in the number of changing variables, which leads to an increase in the computational resources required.

The more-accurate, and this means more-complex, methods of constructing the stress field require a longer time for calculating the problem. The approach, described earlier,[8] enables one to construct the stress field with the same accuracy as the displacements field and

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competes in this sense with the better recovery methods, but exceeds them with respect to the execution time. The basic propositions of this approach are briefly described below and the special features of its actual application are demonstrated taking the example of a plane problem in the linear theory of elasticity.

1. The stress recovery procedure

The essence of the stress recovery procedure is as follows. We write the variational formulation of a boundary value problem in the linear mechanics of a deformable solid in the Lagrange form

$$\int_S \mathbf{q}\cdot\delta\mathbf{u}dS + \int_V \rho\mathbf{K}\cdot\delta\mathbf{u}dV - \int_V \mathbf{T}\cdot\cdot\delta\mathbf{e}dV = 0 \tag{1.1}$$

Here, $\mathbf{q}$ and $\mathbf{K}$ are the surface and mass force vectors, $\mathbf{u}$ is the displacements vector, $\mathbf{e} = (\nabla\mathbf{u} + \nabla\mathbf{u})^T)/2$ is the small deformation tensor (∇ is the Hamiltonian operator in the initial configuration), $\mathbf{T}$ is the stress tensor and ρ is the mass density. We denote the radius vector of the position of a point in the initial state by r, assuming that r depends on the generalized coordinates $q^i : \mathbf{r} = \mathbf{r}(q^i)$. Then, introducing the main basic vectors $\mathbf{r}_i = \partial\mathbf{r}/\partial q^i_r$, constructing the reciprocal basis vectors r^i and taking account of the fact that $\nabla = \mathbf{r}^i(\partial/\partial q^i)$, the convolution in the last integral can be represented in the form

$$\mathbf{T}\cdot\cdot\delta\mathbf{e} = \mathbf{T}\cdot\cdot\nabla(\delta\mathbf{u}) = T^{is}\mathbf{r}_i\mathbf{r}_s\cdot\cdot\mathbf{r}^k\frac{\partial(\delta\mathbf{u})}{\partial q^k} = T^{is}\mathbf{r}_i\cdot\frac{\partial(\delta\mathbf{u})}{\partial q^s} \tag{1.2}$$

Numerical realization of the variational equation (1.1) is achieved using the finite element method (see, for example Ref. [3]). We approximate the displacements vector $\mathbf{u}$ in the domain V, using its values $\mathbf{u}_k$ at the mesh point k (global numbering) and functions of the form $\psi^k_{(j)}$ for the same k-th mesh point of the j-th element are finite functions, the carrier of which is the volume (area) of the finite element (FE) with the number j, that is $\boldsymbol{V}_{(j)}$:

$$\mathbf{u} = \sum_{k=1}^{n}\mathbf{u}_k\left(\sum_{\substack{j=1\\ j\in M_k^V}}^{m}\psi^k_{(j)}\right) \tag{1.3}$$

where M_k^V is the set of numbers of the FEs in the volume V containing the k-th mesh point and n and m are the number of mesh points and FEs. Within the limits of this approximation

$$\frac{\partial(\delta\mathbf{u})}{\partial q^s} = \sum_{k=1}^{n}\delta\mathbf{u}_k\left(\sum_{\substack{j=1\\ j\in M_k^V}}^{m}\psi^k_{(j),s}\right) - \Gamma^l_{sp}\mathbf{r}^p\mathbf{r}_l\cdot\sum_{k=1}^{n}\delta\mathbf{u}_k\left(\sum_{\substack{j=1\\ j\in M_k^V}}^{m}\psi^k_{(j)}\right)$$

where $\psi^k_{(\psi j),s} = \partial\psi^k_{(j)}/\partial q^s$ and Γ^l_{sp} is the Christoffel symbol of the second kind. As a result, when this equality and expression (1.2) are taken into account together with the fact that the variations in the displacements at the mesh points are independent, relation (1.1) reduces to the system of vector equations

$$\sum_{\substack{j=1\\ j\in M_k^S}}^{m}\int_{S_{(j)}}\mathbf{q}\psi^k_{(j)}dS_{(j)} + \sum_{\substack{j=1\\ j\in M_k^V}}^{m}\left(\int_{V_{(j)}}\rho\mathbf{K}\psi^k_{(j)}dV_{(j)} - \int_{V_{(j)}}(T^{is}\psi^k_{(j),i} - T^{il}\Gamma^s_{il}\psi^k_{(j)})\mathbf{r}_s dV_{(j)}\right) = 0,\ k = 1,2,\ldots,n \tag{1.4}$$

Here, M_k^S is the set of numbers of the FEs adjacent to the k-th mesh point with sides belonging to the surface ($M_k^S \subset M_k^V$). For example, in Fig. 1, where subdivision of the body into FEs with global numbering of the mesh points and FEs (in brackets) is shown, the sets M_k^V and M_k^S for the mesh points $k = 3$ and $k = 7$ consist of the FEs with the following numbers

$$M_3^V = \{2,3,8\},\quad M_3^S = \{2,3\},\quad M_7^V = \{1,2,7,8,11,12,16\},\quad M_7^S = \{\varnothing\}$$

Here, $\{\varnothing\}$ is an empty set.

Any k-th equation of system (1.4) can be treated as the sum of the forces from each FE containing the mesh point k reduced to this mesh point. Determining the components of the stress tensor T^{is} in system (1.4) using Hooke's law, we obtain a system of linear algebraic equations in the displacements at the mesh points, and, the solving this, we find the values of the displacements vector $\mathbf{u}_k$ at the mesh points of the FEs.

We now select a fairly smooth surface l within the body, formed by the sides of the FE and dividing the body into two parts (see Fig. 1). The numbers of the mesh points belonging to the surface l form the set N_l. Here,

$$N_l = \{4,8,13,19\}$$

If one part of the body is discarded, then the vector of the unknown distributed force p will be its force interaction with the remaining part of the body bounded by the surface $S_* = S_1 \cup l$. Kinematic and force boundary conditions are specified on the surface S_1. Within the

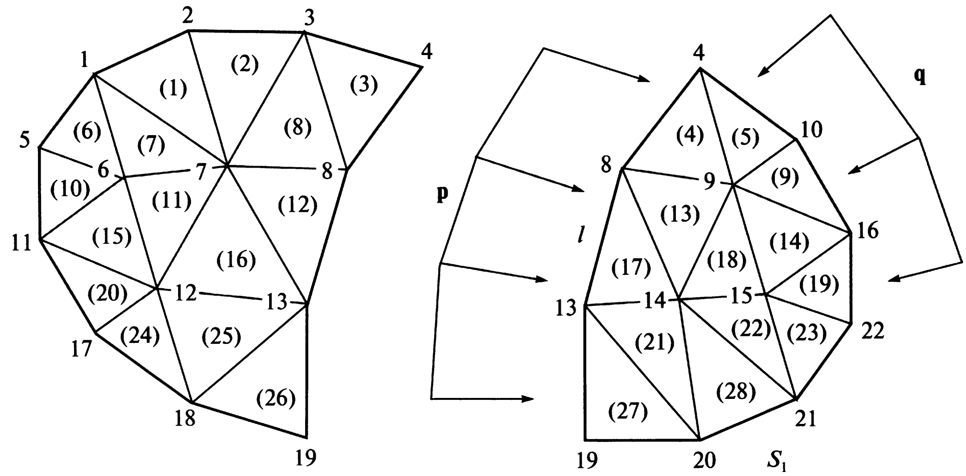

Fig. 1.

volume V_* bounded by the surface S_*, the mesh point values of the displacements vector $\mathbf{u}_k$ are known from the solution of system (1.4). The distributed force on the surface l is an unknown quantity.

Writing the variational equation for the domain V_* and implementing the finite element method for realizing of this equation, we arrive at the following system

$$\sum_{\substack{j=1\\ j\in M_k^{S_*}}}^{m}\int_{S_{(j)}}\mathbf{q}_*\psi_{(j)}^k dS_{(j)}+\sum_{\substack{j=1\\ j\in M_k^{V_*}}}^{m}\left(\int_{V_{(j)}}\rho\mathbf{K}\psi_{(j)}^k dV_{(j)}-\int_{V_{(j)}}(T^{is}\psi_{(j),i}^k-T^{il}\Gamma_{il}^s\psi_{(j)}^k)\mathbf{r}_s dV_{(j)}\right)=0,\ k\in N_* \tag{1.5}$$

where $M_k^{V_*}$ is the set of numbers of the FEs belonging to the k-th mesh point in the volume V_*, $M_k^{S_*}$ is the set of numbers of the FEs belonging to the k-th mesh point with sides belonging to the surface $S_*(M_k^{S_*}\subset M_k^{V_*})$, N_* is the set of numbers of the mesh points of the FEs in the volume V_*, and q_* are the distributed forces which are given on S_1 ($\mathbf{q}_*=\mathbf{q}$) or unknown on the surface l ($\mathbf{q}_*=\mathbf{p}$). In the case of the mesh points belonging to the domain $V_*\cup S_1$, Eqs. (1.5), which are a part of system (1.4), are identically satisfied for the values of the mesh point values of the displacements vectors $\mathbf{u}_k$ obtained by solving system (1.4). Actually, for these mesh points, the sets $M_k^{V_*}$ and $M_l^{S_*}$ are identical to the sets M_k^V and M_k^S respectively. Equations (1.4) and (1.5) are therefore identical for these mesh points. In the case of mesh points belonging to the surface l, the equations of system (1.5) are represented in the form

$$\sum_{\substack{j=1\\ j\in M_k^{l}}}^{m}\int_{l}\mathbf{p}\psi_{(j)}^k dl=\mathbf{Q}_k-\sum_{\substack{j=1\\ j\in M_k^{S_1}}}^{m}\int_{S_1}\mathbf{q}\psi_{(j)}^k dS_1,\quad k\in N_l \tag{1.6}$$

where $M_k^{S_1}$ and M_k^l are the sets of numbers of the FEs belonging to the k-th mesh point with sides belonging to the surfaces S_1 and l respectively, N_l is the set of the numbers of the mesh points belonging to the surface l, and

$$\mathbf{Q}_k=\sum_{\substack{j=1\\ j\in M_k^{V_*}}}^{m}\left(\int_{V_{(j)}}(T^{is}\psi_{(j),i}^k-T^{il}\Gamma_{il}^s\psi_{(j)}^k)\mathbf{r}_s dV_{(j)}-\int_{V_{(j)}}\rho\mathbf{K}\psi_{(j)}^k dV_{(j)}\right)$$

are the force vectors reduced to the k-th mesh point. These latter quantities are known since they are determined for each FE $V_{(j)}$ using the mesh point values of the displacements vectors $\mathbf{u}_k$ obtained by solving system (1.4).

System (1.6) is a system of Fredholm equations of the first kind which defines the vector of the unknown distributed forces $\mathbf{p}$ on the surface l. Fredholm integral equations of the first kind belong to the class of problems which are ill-posed in the Hadamard sense, and Tikhonov regularizers[9] will be used to solve them.

In a similar manner to the approximation (1.3), we approximate the vector $\mathbf{p}$ using its mesh point values $\mathbf{p}_k$ and the same function of the form $\psi_{(j)}^k$ as in representation (1.3):

$$\mathbf{p}=\sum_{k\in N_l}\mathbf{p}_k\sum_{j\in M_k^l}\psi_{(j)}^k \tag{1.7}$$

that is, the vectors $\mathbf{u}$ (1.3) and $\mathbf{p}$ (1.7) have the same order of approximation. We use the method of least squares and minimize the functional

$$\min_{\mathbf{p}_k}\left[\int_l\left(\mathbf{p}-\sum_{k\in N_l}\mathbf{p}_k\sum_{j\in M_k^l}\psi_{(j)}^k\right)\cdot\left(\mathbf{p}-\sum_{k\in N_l}\mathbf{p}_k\sum_{j\in M_k^l}\psi_{(j)}^k\right)dl+\alpha R(\mathbf{p}_k)\right] \tag{1.8}$$

with respect to p_k. Here, $R(\mathbf{p}_k)$ is a Tikhonov regularizer with parameter α. $R_{(\mathbf{p}_k)}$ can be a zero order regularizer

$$R_0(\mathbf{p}_k) = \sum_{k\in N_l} \mathbf{p}_k \cdot \mathbf{p}_k$$

which imposes a condition of smallness on the modulus of the vector p_k, or a first order regularizer

$$R_1(\mathbf{p}_k) = \sum_{k,i\in N_l,\, i\neq k} \beta_{ki}(\mathbf{p}_k - \mathbf{p}_i)\cdot(\mathbf{p}_k - \mathbf{p}_i) \tag{1.9}$$

which imposes a condition of smallness on the modulus of the difference between the vectors $\mathbf{p}_k$ and $\mathbf{p}_i$. Here, the weighting factors β_{ki} decrease according to some law as the distance between the mesh points k and i increases. It is often convenient to assume that, if the mesh points k and i belong to different FEs, then $\beta_{ki} = 0$ and, if they belong to the one and the same mesh point, then $\beta_{ki} = 1$.

Minimization of functional (1.8) leads to the system of equations

$$\sum_{j\in M_k^l} \int_l \mathbf{p}\psi_{(j)}^k dl = \sum_{j\in M_k^l} \int_l \psi_{(j)}^k \sum_{i\in N_j} \mathbf{p}_i \psi_{(j)}^i dl + \begin{cases} \alpha\mathbf{p}_k & \text{для } R_0(\mathbf{p}_k) \\ \alpha \sum\limits_{i\in N_{l,i\neq k}} \beta_{ki}(\mathbf{p}_k - \mathbf{p}_i) & \text{для } R_1(\mathbf{p}_k) \end{cases}, \; k\in N_l \tag{1.10}$$

Here N_j is the set of number of the mesh points belonging to FE j and account has been taken of the orthogonality of functions of the form $\psi_{(j)}^k$ and $\psi_{(s)}^i$ when $j \neq s$. System (1.10) has a symmetric tridiagonal matrix in the case of a linear approximation of the displacements field and a pentadiagonal matrix in the case of a quadratic approximation. Replacing the left-hand side of equality (1.10) in accordance with equality (1.6), we obtain a system of linear algebraic equations which enable us to find the values of $\mathbf{p}_k$ on the surface l.

In order to determine the components of the stress tensor, we write the Cauchy relations for the mesh point k on the surface l

$$\mathbf{n}_k^l \cdot \mathbf{T}_k = \mathbf{p}_k^l \tag{1.11}$$

Here $\mathbf{n}_k^l$ is the outward unit normal to the surface l at the mesh point k, $\mathbf{T}_k$ is the value of the stress tensor $\mathbf{T}$ at this mesh point. In the case of a two-dimensional problem, the tensor $\mathbf{T}_k$ is defined by three independent components, that is, system (1.11) is sub-definite (a system of two equations with three unknowns). In order to find all the components of the tensor $\mathbf{T}_k$, we consider yet another fairly smooth surface t containing the mesh point k. Determining $\mathbf{p}_k$ on the surface t with an outward unit normal $\mathbf{n}_k^t$ and writing the Cauchy relation for the mesh point k on the surface t

$$\mathbf{n}_k^t \cdot \mathbf{T}_k = \mathbf{p}_k^t \tag{1.12}$$

we obtain the overdefined system of equations (1.11) and (1.12) (a system of four equations with three unknowns) which can be solved by the method of least squares.

2. The use of the stress recovery procedure to solve a plane problem of the linear theory of elasticity

We will now demonstrate the use of the stress recovery procedure by solving a problem in the linear theory of elasticity concerning the uniaxial elongation of a square plate, which is under conditions of plane strain (a section of an infinitely long rod), by forces (unlike the problem considered earlier[8]) applied to its two ends. In view of the symmetry of the problem, we will consider a quarter of the plate (Fig. 2) with the following boundary conditions

$$\begin{aligned} &x = 0: \;\; u_x = 0, \;\; T_{xy} = 0; \;\; y = 0: \;\; u_y = 0, \;\; T_{xy} = 0 \\ &x = a: \;\; T_{xx} = 0, \;\; T_{xy} = 0; \;\; y = a: \;\; T_{yy} = p_y(x), \;\; T_{xy} = 0 \end{aligned} \tag{2.1}$$

Here, u_x, u_y, p_y and T_{xx}, T_{xy}, T_{yy} are components of the displacements vector $\mathbf{u}$, the distributed force vector $\mathbf{p}$ and the stress tensor T in a Cartesian system of coordinates. Finite element discretization of the calculated region is carried out using an $n\times m$ mesh with triangular FEs. We will consider horizontal layers ($i = 1, 2, \ldots m$) and vertical layers ($j = 1, 2, \ldots, n$). In the horizontal layers, we isolate the lower surface A and the upper surface B, and, in the vertical layers, the left surfaces, C and right surfaces, D. For each of these surfaces, we introduce a local numbering of the mesh points from 1 to $n+1$ (Arabic numbers) for the horizontal layers and from I to m + 1 (Roman numbers) for the vertical layers. We denote the parts of a surface adjacent to the mesh point k by $(k-1)$ and (k) (concerning everything which has been said above, see Fig. 2).

In the first stage, we solve the initial problem using the finite element method; multiplying the mesh point values of the displacements vector $\mathbf{u}_k$ obtained by the stiffness matrices (which have been formed for the horizontal and vertical layers), we obtain the force vectors reduced to the mesh points. In the second stage, using these forces at the mesh points we find the forces p_x and p_y distributed over the surfaces of the horizontal and vertical layers.

We now consider the horizontal layer $i = 1$. According to boundary conditions (2.1), $p_x = 0$ on the surface A of this layer. The remaining unknown quantity p_y on this surface is represented within the limits of the linear approximation and the local numbering of the mesh points, which has been discussed above in this section, as

$$p_y = p_1^y\psi_{(1)}^1 + p_2^y(\psi_{(1)}^2 + \psi_{(2)}^2) + \cdots + p_n^y(\psi_{(n-1)}^n + \psi_{(n)}^n) + p_{n+1}^y\psi_{(n)}^{n+1}$$

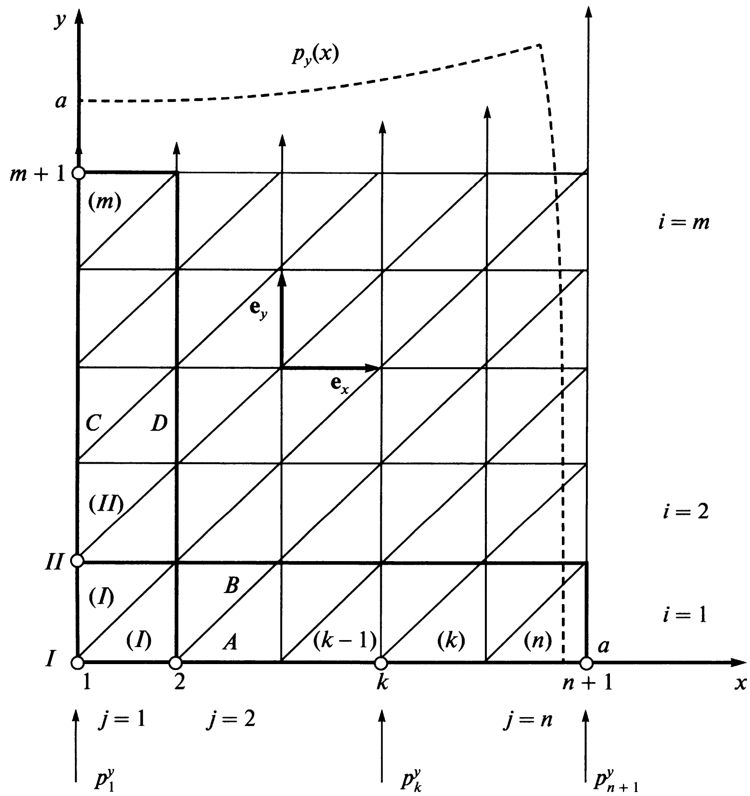

Fig. 2.

Then, functional (1.8) with regularizer (1.9) when $\beta_{ki} = 1$ takes the form

$$\min_{p_k^y}\left[\int_0^a (p_y - [p_1^y\psi^1_{(1)} + p_2^y(\psi^2_{(1)} + \psi^2_{(2)}) + \dots + p_n^y(\psi^n_{(n-1)} + \psi^n_{(n)}) + p_{n+1}^y\psi^{n+1}_{(n)}])^2 dx + \right.$$
$$\left. + \alpha[(p_1^y - p_2^y)^2 + (p_2^y - p_3^y)^2 + \dots + (p_n^y - p_{n+1}^y)^2]\right]$$

and its minimization leads to the following system of equations

$$\begin{aligned}
&p_1^y(\Psi^{1,1}_{(1)} + \alpha) + p_2^y(\Psi^{1,2}_{(1)} - \alpha) = \Phi^1_{(1)}\\
&\dots\dots\dots\dots\dots\\
&p_1^y(\Psi^{1,2}_{(1)} - \alpha) + p_2^y(\Psi^{2,2}_{(1)} + \Psi^{2,2}_{(2)} + 2\alpha) + p_3^y(\Psi^{2,3}_{(2)} - \alpha) = \Phi^2_{(1)} + \Phi^2_{(2)}\\
&p_{n-1}^y(\Psi^{n-1,n}_{(n-1)} - \alpha) + p_n^y(\Psi^{n,n}_{(n-1)} + \Psi^{n,n}_{(n)} + 2\alpha) + p_{n+1}^y(\Psi^{n,n+1}_{(n)} - \alpha) = \Phi^n_{(n-1)} + \Phi^n_{(n)}\\
&\dots\dots\dots\dots\dots\\
&p_n^y(\Psi^{n,n+1}_{(n)} - \alpha) + p_{n+1}^y(\Psi^{n+1,n+1}_{(n)} + \alpha) = \Phi^{n+1}_{(n)}
\end{aligned} \tag{2.2}$$

Here,

$$\Psi^{r,s}_{(k)} = \int_{(k)} \psi^r_{(k)}\psi^s_{(k)}dx, \quad \Phi^s_{(k)} = \int_{(k)} p_y\psi^s_{(k)}dx$$

On the other hand, from relations (1.6), we have the following identities for $\Phi^S_{(k)}$

$$\Phi^1_{(1)} \equiv Q^y_1 - \int\limits_{(I)} p_y \Psi^I_{(I)}\,|_{x=0}\, dy, \quad \Phi^{n+1}_{(n)} \equiv Q^y_{n+1} - \int\limits_{(I)} p_y \Psi^I_{(I)}\,|_{x=a}\, dy$$

$$\Phi^s_{(s-1)} + \Phi^s_{(s)} \equiv Q^y_s, \quad s = 2,\dots,n$$

where Q^y_k is the force in the y direction at the mesh point k (obtained in the first stage of solving the problem).

Hence, a system of linear algebraic equations with symmetric positive definite tridiagonal matrix (2.2) is written for the quantities p^y_k on the surface A of the layer $i=1$ (we obtain a pentadiagonal matrix in the case of the quadratic approximation of the quantity p_y). Since, according to conditions (2.1), $p_y=0$ when $x=0$ and $x=a$, the right-hand sides in the first and last relations of system (2.2) are equal to Q^y_1 and Q^y_{n+1} respectively.

In determining p^x_k and p^y_k on the surface B of the horizontal layer i and on the surface C of the vertical layer j, we obtain systems which are similar to system (2.2). Apart from the known forces at the mesh points Q^x_k and Q^y_k, integrals of the distributed forces p_x and p_y will occur on the right-hand side of the first and last equations. If these forces are not determined by the boundary conditions of the problem, they have to be found in advance. There must therefore be a strategy for calculating the distributed forces on the surfaces of the horizontal and vertical layers.

In the above problem, this strategy consists of the following:

1) we find p_y on the surface A when $i=1$ and on the surface B when $i=1, 2, \dots, m$ using a system of the form of (2.2) and taking account of the fact that $p_y=0$ when $x=0$ and $x=a$;
2) we determine p_x on the surfaces C when $j=1, 2, \dots, n$, taking account of the fact that $p_x=0$ when $y=0$ and $y=a$;
3) we calculate p_y on the surfaces C when $j=1, 2, \dots, n$, taking account of the fact that, when $y=0$, p_y is a known quantity after carrying out step 1 and, when $y=a$, p_y is specified by the boundary conditions;
4) we construct p_x on the surfaces B when $i=1, 2, \dots, m$ taking account of the fact that, when $x=0$, p_x is a known quantity after step 2 and, when $x=a$, $p_x=0$ according to the boundary conditions.

Note that the values of the quantities p_x and p_y on the surface D when $j=n$ are specified by the boundary conditions.

Knowing the forces p_x and p_y on all the surfaces and using the Cauchy relation $\mathbf{n}\cdot\mathbf{T}=\mathbf{p}$, we calculate the components of the stress tensor T_{xx}, T_{xy} and T_{yy}. We now consider a certain mesh point k at which the horizontal surface l intersects the vertical surface t. Suppose $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors of the x and y axes (basis vectors, see Fig. 2). Then,

$$\mathbf{T} = T_{xx}\mathbf{e}_x\mathbf{e}_x + T_{xy}(\mathbf{e}_x\mathbf{e}_y + \mathbf{e}_y\mathbf{e}_x) + T_{yy}\mathbf{e}_y\mathbf{e}_y, \quad \mathbf{p} = p_x\mathbf{e}_x + p_y\mathbf{e}_y$$

Taking account of the fact that $\mathbf{n}^l_k = \mathbf{e}_y$ for the surface l which coincides with B and that $\mathbf{n}^t_k = \mathbf{e}_x$ for the surface t which coincides with D, we rewrite relations (1.11) and (1.12) in the form

$$T_{xy}\mathbf{e}_x + T_{yy}\mathbf{e}_y = p^l_x\mathbf{e}_x + p^l_y\mathbf{e}_y, \quad T_{xx}\mathbf{e}_x + T_{xy}\mathbf{e}_y = p^t_x\mathbf{e}_x + p^t_y\mathbf{e}_y$$

It follows from this that

$$p^l_x = T_{xy}, \quad p^l_y = T_{yy}, \quad p^t_x = T_{xx}, \quad p^t_y = T_{xy} \tag{2.3}$$

and that the stress T_{xy} can be determined from the two relations. When solving the problem, the cases when $T_{xy} = p^l_x$ and $T_{xy} = p^t_y$ were considered.

The specific values of the quantities in the problem being solved were taken to be the following. The length of a side of the square plate (sections of an infinitely long rod) was assumed to be equal to unity: $a=1$. The behaviour of the material was described by Hooke's law $\mathbf{T}=\lambda I_1(\mathbf{e})\mathbf{g}+2G\mathbf{e}$ ($\mathbf{g}$ is the unit tensor and $I_1(\mathbf{e})$ is the first invariant of the strain tensor) with constants $\lambda=1.5$ and $G=1$. The force on the ends was specified in the form of the function $p_y(x)=0.1(1+5x^4)$. The problem was solved on $n\times m=5\times 5$, 10×10, 20×20, 40×40 meshes for the linear, quadratic and cubic approximation of the strain field (unlike the approach adopted earlier[8]). The stress fields were determined by differentiation of the strain fields obtained (the conventional method), reducing the order of the approximation of the first of these compared with the last by unity. Here, the discontinuous stress field, corresponding to the linear approximation of the strain field was reduced to the mesh point by the usual method of averaging over the elements adjacent to it (the stress fields in the quadratic and cubic approximations of the strain field are continuous). Furthermore, the stress fields were constructed for each of the above mentioned approximations of the strain fields using the recovery procedure described above for the regularization parameters $\alpha=10^{-6}$ and $\alpha=1$.

The contour of the deformed plate is shown (for information) by the dashed line in Fig. 2. In the case of the internal points of the region, as previously,[8] the application of the procedure developed to the solution obtained for the linear (quadratic) approximation of the strain field enabled us to construct the stress field with the same accuracy which the conventional method of differentiating the strain field gives, but for the solution obtained with the quadratic (cubic) approximation of the strain field. Hence, the strain and stress fields in the procedure developed are of the same order of approximation. This is illustrated in Fig. 3 using the example of the T_{xy} component of the stress tensor. For the boundary of the region where the forces (stresses) are given, use of the stress recovery procedure, unlike the other methods, gives accurate values of these forces (stresses) for any approximation of the strain field.

In accordance with equality (2.3), the stresses $T_{xy} = p^l_x$ are calculated on the horizontal surfaces passing through the mesh points and the same stresses $T_{xy} = p^t_y$ are calculated on the vertical surfaces. The stress distribution T_{xy} in the section $y=0.5$ is shown in Figs. 3 and 4 for a 40 H 40 mesh. Curve *1* in Fig. 3 was obtained by the usual method for the linear approximation of the strain field and curve *2* for the quadratic approximation. The stress distribution constructed by the recovery method when $\alpha=10^{-6}$ in the linear approximation of the

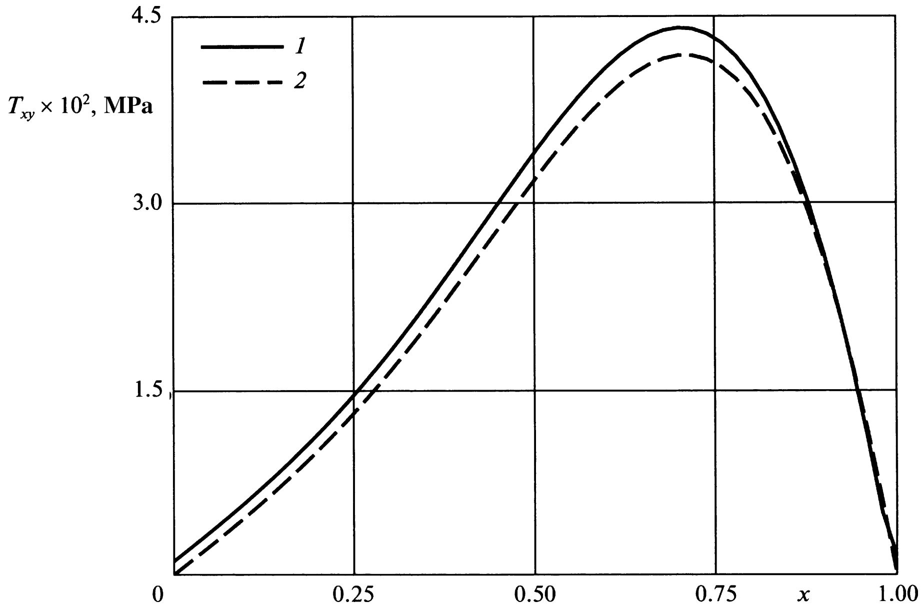

Fig. 3.

strain field (the calculation was carried out on the vertical surfaces ($T_{xy} = p_y^t$)) was practically identical to curve *2*. The values of the stresses T_{xy} at the boundaries of the region $x = 0$ and $x = a$ must be zero. As a result of the calculation, it was found that, when $x = 0$, $T_{xy} = 1 \cdot 10^{-3}$ MPa for curve *1* and $T_{xy} = -9.9 \cdot 10^{-5}$ MPa for curve 2, and, when $x = a$, 1.5 H 10^{-3} MPa and 2.7 H 10^{-5} MPa. The values of T_{xy} calculated by the stress recovery method are exactly equal to zero on these boundaries.

Curve *1* in Fig. 4 corresponds to the shear stress found using the stress recovery method on the vertical surfaces when $\alpha = 10^{-1}$ (as in Fig. 3), and curves *2* and *3* were found by the same method on the horizontal surfaces when $\alpha = 10^{-6}$ and $\alpha = 1$ respectively. The approximation of the displacements field here is the same linear approximation. Curve *2* has considerable oscillations of the edges of the plate, which is characteristic of the Gibbs effect. An increase in α leads to a decrease in the oscillations (curve *3*) but does not lead to zero values of T_{xy} on the vertical surface of the plate.

In concluding, we note that the stress recovery procedure described enables one to construct the stress field with the same accuracy (of the same order of approximation) as the displacements field.

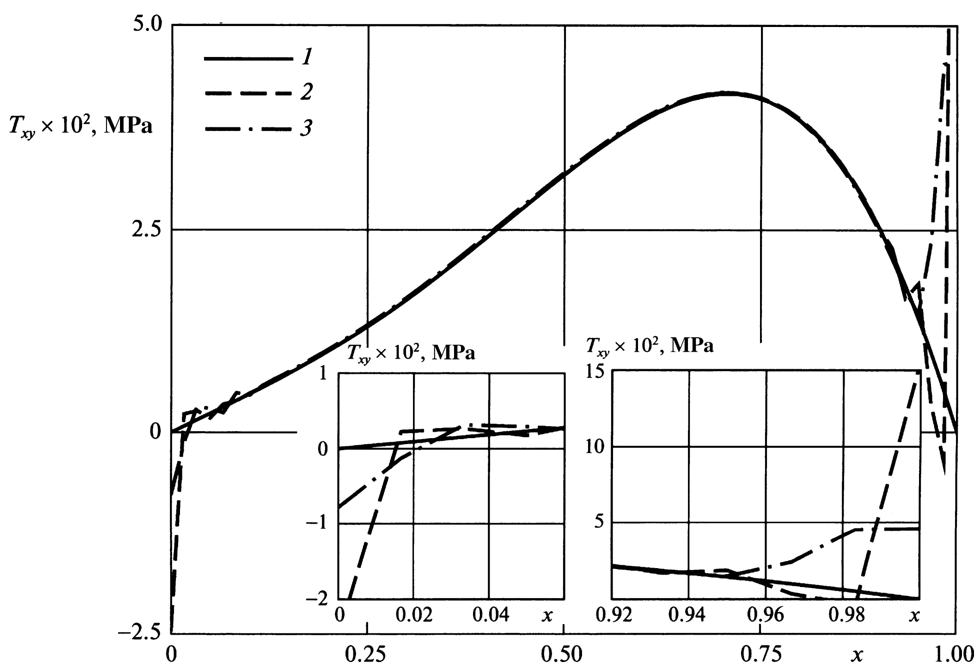

Fig. 4.

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